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ON THE GLOBAL STABILITY OF THE FORCED MOTIONS OF A LIQUID WITHIN AN ELLIPSOID*

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Under certain conditions the rotation of the liquid in a liquid gyroscope around the central axis of an ellipsoid loses stability /1-4/. This occurs, for example, in the case when the moment of the external forces directed along the central axis exceeds a certain critical (bifurcational) value. The aim of this paper is to obtain the sufficient conditions for global asymptotic stability /5/ which, at the same time, give rise to stationary motions of the liquid around the principal axes of the ellipsoid. In solving this problem, use is made of the ideas, methods and results described in /1, 2, 5-9/**.(**See also Yudovich V.I., Asymptotic form of the limiting cycles of a Lorenz system at large Rayleigh numbers. Rostov-on-Don, 1977: Deposited in the All-Union Inst. for Scientific and Technical Information (VINIT), 2611-78, 31.07.78.)

We know that, when there is a sufficiently strong action on the central axis of a gyroscope, there are two stationary rotations which differ in the directions of the liquid flow around the stable axes /1, 8/. A theoretical treatment of the stability of the motions has only previously been carried out when there are small deviations from the stationary flows /1-4, 10, 11/, that is, the Lyapunov stability or the stability under the assumption that the constant external and dissipative moments were small /12/ has been analysed. The global asymptotic stability of the above-mentioned stationary motions is investigated below. The term "global asymptotic stability" is analogous to the term "stability on the whole" and is employed in the analysis of systems with a non-unique equilibrium position /5, 13/.

Let us consider the system of equations

$$\sigma^{\cdot} = \eta,\ \eta^{\cdot} = -\psi(\sigma, \eta) - f(\sigma)z - \varphi(\sigma) \tag{1}$$
$$z^{\cdot} = -Az - g(\sigma)\eta;\ A = \text{const}$$

containing continuously differentiable functions and where $\psi(\sigma, \eta)\eta \geqslant \mu\eta^2$, $\forall\sigma \in R^1$ for certain $\mu > 0$.

Many problems in the theory of synchronous machines /7, 14/ and of non-linear oscillators with an automatic regulator /8, 15/ lead to an analysis of equations of the form of (1). It is well-known that a Lorenz system as well as a liquid gyroscope system which is forced along an unstable axis can be reduced to system (1) /6, 8/. The Lorenz model also describes the motion of a "liquid gyroscope" in the field of Coriolis forces in the case when the ellipsoid containing the liquid is an ellipsoid of rotation /11/.

We shall say that system (1) is globally and asymptotically stable if any solution of this system $X(t) = \text{col}\{\sigma(t), \eta(t), z(t)\}$ tends, as $t \to +\infty$, to a certain equilibrium position /5/, while it is **stable as a whole if it** has a single equilibrium position and is globally asymptotically stable /13/ and dissipative in the Levinson sense, if a bounded, closed domain of attraction exists in the phase space σ, η, z and z /16/. Systems of the hydrodynamic type /1/ constitute a well-known class of systems which are dissipative according to Levinson and the simplest model of this class is considered below.

We shall assume that the following two conditions are satisfied. 1°. Any solution of system (1) is infinitely extendible to the right (in particular, if system (1) is Levinson-dissipative, this assumption is satisfied). 2°. The stationary set of system (1) /5/ consists of isolated points.

The following theorem generalizes the results of /6/ for system (1).

Theorem 1. If, in the case of a certain number $\alpha > 0$ and a bounded solution $X(t)$ system (1) satisfies the inequality

$$\overline{\lim_{t\to\infty}}\,|\alpha^2 g(\sigma(t)) + f(\sigma(t))| < 2\alpha\sqrt{A\mu} \tag{2}$$

then it **tends** to one of the equilibrium positions.

Proof. Let $X(t, X_0)$ be a bounded solution of system (1) which satisfies inequality (2) for a certain $\alpha > 0$. Let us consider the Lyapunov function

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$$W(X) = \frac{\alpha^2}{2} z^2 + \frac{1}{2}\eta^2 + \int\limits_0^\sigma \varphi(\sigma)\, d\sigma$$

By virtue of system (1), the estimate

$$W^{\cdot} = \alpha^2 z[-Az - g(\sigma)\eta] + \eta[-\psi(\sigma,\eta) - f(\sigma)z - \varphi(\sigma)] + \varphi(\sigma)\eta \leqslant$$
$$-\left[(\alpha\sqrt{A}\,z)^2 + 2\,\frac{[\alpha^2 g(\sigma) + f(\sigma)}{2\alpha\sqrt{A\mu}}\,\alpha\sqrt{A\mu}\,z\eta + (\sqrt{\mu}\,\eta)^2\right]$$

holds for an arbitrary function $W(X)$.

It can be seen from this that $W^{\cdot}(X) < 0$ when inequality (2) is satisfied. Then, on the trajectory $X(t, X_0)$ of system (1), the function $W(X(t, X_0))$ does not increase with respect to t in a certain interval $(\tau, +\infty)$. It follows from this and from the boundedness of $W(X(t, X_0))$ that a finite limit $\lim W(X(t, X_0)) = m$ exists as $t \to +\infty$. The subsequent proof repeats exactly the proof of the theorem in /6/.

The theorem which has been proved in conjunction with the estimates of the domains of dissipation leads to the conditions for global asymptotic stability.

Let us now consider the system of transformed equations for a forced liquid gyroscope allowing for disspation /1, 8/

$$\begin{gathered} x_1^{\cdot} = y_1^2 - z_1^2 - lx_1 + F_0 \\ y_1^{\cdot} = -x_1 y_1 - l y_1, \quad z_1^{\cdot} = x_1 z_1 - l z_1 \end{gathered} \tag{3}$$

Here F_0 is the constant moment of the external forces acting along the central axis of the ellipsoid and l is the dissipation parameter (the friction is assumed to be isotropic).

Let us now introduce a new variable, the energy of the system

$$z_0 = {}^1/_2\,(x_1^2 + y_1^2 + z_1^2) \tag{4}$$

into the treatment.

Differentiating (4), by virtue of system (3), we have

$$z_0^{\cdot} = -2lz_0 + F_0 x_1 \tag{5}$$

By differentiating the first equation of system (3) and making use of the two other equations and the function (4), we obtain

$$x_1^{\cdot\cdot} = 2x_1^3 - (4z_0 + 2l^2)x_1 - 3lx_1^{\cdot} + 2lF_0 \tag{6}$$

Now, by putting

$$x_1 = \sigma, \quad x_1^{\cdot} = \sigma^{\cdot} = \eta, \quad z_0 = z + {}^1/_2 F_0 l^{-1}\sigma$$

in (5) and (6), we arrive at system (1), where

$$\begin{gathered} \psi(\sigma,\eta) = 3l\eta, \quad A = 2l, \quad g(\sigma) = {}^1/_2 F_0 l^{-1} \\ f(\sigma) = 4\sigma, \quad \varphi(\sigma) = 2l^{-1}(l^2 - \sigma^2)(l\sigma - F_0) \end{gathered}$$

By putting $\mu = 3l$, we arrive at the following assertion.

Corollary 1. It follows from Theorem 1 that, if a bounded solution of system (3) satisfies the condition

$$a_- < \varliminf_{t\to\infty} \sigma(t) < \varlimsup_{t\to\infty} \sigma(t) < a_+, \quad a_\pm = \pm\frac{\sqrt{6}}{2}\,l\alpha - \alpha^2\frac{F_0}{8l} \tag{7}$$

then it tends to one of the equilibrium positions.

By means of the substitution (similar to the substitution in /1/)

$$x_1 = x + F_0 l^{-1}, \quad y_1 = {}^1/_2\,(z + y), \quad z_1 = {}^1/_2\,(z - y)$$

we reduce system (3) to the form

$$\begin{gathered} x^{\cdot} = -lx + yz \\ y^{\cdot} = -ly - F_0 l^{-1} z - xz, \quad z^{\cdot} = -lz - F_0 l^{-1} y - xy \end{gathered} \tag{8}$$

When $F_0 \leqslant l^2$, system (8) has a single equilibrium position $C_0(0, 0, 0)$ which corresponds to a unique stationary motion of the liquid. When $F_0 > l^2$, there are three stationary motions: C_0 and $C_{1,2}\,(l - F_0 l^{-1}, \pm\sqrt{F_0 - l^2}, \mp\sqrt{F_0 - l^2})$.

Stationary motions **which differ** in their directions of rotation around the stable axes physically **correspond** to the equilibrium positions C_1 and C_2 /1/.

In order to prove the **dissipation** properties of system (8) and to obtain actual estimates of its domain, we introduce the function V and the number Γ into the treatment:

$$V = x^2 + \frac{y^2 + z^2}{2} + \theta x, \quad \Gamma = \frac{(l - 2\lambda)^2\theta^2}{16\lambda(l - \lambda)}$$
$$(\theta l = 2[F_0 - l(l - \lambda)], \ \lambda \in (0, l))$$

Lemma 1. the inequality

$$\overline{\lim_{t\to\infty}} V(x(t), y(t), z(t)) \leqslant \Gamma \tag{9}$$

holds for any solution of system (8).

Proof. We have

$$V^{\cdot} + 2\lambda V = -2(l-\lambda)\left[x + \frac{(l-2\lambda)\theta}{4(l-\lambda)}\right]^2 + \frac{(l-2\lambda)^2\theta^2}{8(l-\lambda)} - (l-\lambda)(y+z)^2 \leqslant 2\lambda\Gamma$$

Let us represent this inequality in the form

$$(V-\Gamma)^{\cdot} e^{2\lambda t} + 2\lambda(V-\Gamma)e^{2\lambda t} \leqslant 0$$

After integration from 0 to t_1, we obtain

$$V(x(t_1), y(t_1), z(t_1)) - \Gamma \leqslant [V(x(0), y(0), z(0)) - \Gamma]e^{-2\lambda t_1}$$

The latter holds for any $t_1 \geqslant 0$ which proves the assertion of the lemma.

Corollary 2. It follows from inequality (9) that the estimate

$$b_- \leqslant \underline{\lim_{t\to\infty}} x(t) \leqslant \overline{\lim_{t\to\infty}} x(t) \leqslant b_+, \quad b_\pm = -\frac{\theta}{2} \pm \frac{\theta l}{4\sqrt{\lambda(l-\lambda)}} \tag{10}$$

holds for any solution of system (8).

By comparing (7) and (10) and, at the same time, putting $\alpha = 2\sqrt{2}, \lambda = l/2$, we arrive at the following assertion.

Corollary 3. If the moment of the external forces F_0 and the dissipation parameter l satisfy the inequality $F_0 > \sqrt{3}l^2$, system (3) is globally asymptotically stable.

By introducing the Reynolds number $R = F_0/l^2$ which is defined with respect to the pressure /1/, we finally conclude that, when $R < 1$, the liquid gyroscope system (3) is stable as a whole but, when $1 < R < \sqrt{3}$, it is globally asymptotically stable.

Thus, the simplest three-mode Galerkin approximation of the equations describing the motion of a liquid in an ellipsoidal cavity in the field of an action which is constant along an unstable axis has been considered. It has previousy been proved /1-4/ that, in such a system, non-unique equilibrium positions occur when $R > 1$: one is unstable and two are Lyapunov-stable. However, as we know, the existence of equilibrium positions in a system which are Lyapunov-stable does not preclude the existence of attractors of other types such as, for examples, limiting cycles. The condition obtained here separates out a domain in the parameter space in which the liquid gyroscope system (3) is globally asymptotically stable. When this condition is realized, the physical system, for any initial perturbations, tends, as $t \to +\infty$, to one of the two possible stationary motions.

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REACTION OF A PIEZOCERAMIC SHELL TO CONCENTRATED DYNAMICAL ACTIONS*

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A solution of the equations of motion of an infinite, cylindrical piezoceramic shell, the facial surfaces of which are not covered by electrodes but subjected to a periodic system of concentrated forces varying harmonically with time, is constructed. Green's function method is used for this purpose. In the case of regular roots of the dispersion equation the unique solution is picked out on the basis of limiting absorption principle. The irregular roots determine the spectrum of resonance frequencies. An analytical and numerical analysis of the roots of the dispersion equation is carried out. A qualitative picture of the wave process is given. The results of a calculation of the amplitude-frequency characteristics of the displacement and the electric field potential are presented as well as a comparison with a non-electric shell. The free vibrations of piezoceramic shells are considered in /1, 2/ and the forced vibrations of such shells in /3, 4/.

1. Let us consider a cylindrical piezoceramic shell which is referred to the orthogonal coordinates α, β and z, polarized along the α coordinate and loaded with a system of concentrated forces which are periodic with respect to β and vary harmonically with time. The facial surfaces of the shell are free from electrodes and are bounded by a vacuum. When account is taken of the equations of state /5/, the equations for the steady-state vibrations of such a shell have the form

$$L_{ij}u_j = P_i\delta(\alpha, \beta) + \delta_i\rho h\omega^2 u_i \tag{1.1}$$
$$\delta_1 = \delta_2 = -1,\ \delta_3 = 1,\ \delta_4 = 0,\ i, j = 1, 2, 3, 4$$

Here, $u_j(\alpha, \beta)$ are the amplitudes of the displacements $(j = 1, 2, 3)$, $u_4 = \varphi(\alpha, \beta)$ is the electric field potential in the shell, $\delta(\alpha, \beta)$ is a two-dimensional Dirac function, ω is the frequency, ρ and h are the density of the material and the thickness of the shell and P_i is the amplitude of the corresponding concentrated force.

The differential operators L_{ij} are written out in /6/ and the coefficients occurring in them have the form

$$c_{33} = c_{33}^E[1 - (c_{13}^E)^2/(c_{11}^E c_{33}^E)]$$
$$c_{13} = c_{13}^E[1 - c_{12}^E/c_{11}^E],\quad c_{11} = c_{11}^E[1 - (c_{12}^E/c_{11}^E)^2]$$
$$e_{31} = e_{31}^*[1 - c_{12}^E/c_{11}^E],\ e_{15} = e_{15}^*$$
$$e_{33} = e_{33}^*[1 - e_{31}^* c_{13}^E/(e_{33}^* c_{11}^E)],\quad c_{44} = c_{44}^E$$
$$\varepsilon_{33} = \varepsilon_{33}^S[1 + (e_{31}^*)^2/(c_{11}^E\varepsilon_{33}^S)],\ \varepsilon_{11} = \varepsilon_{11}^S$$

Here, c_{ij}^E are the coefficients of elasticity of the piezoceramic when the electric field is zero, ε_{11}^S and ε_{33}^S are the permittivities when the stresses are zero and e_{ij}^* are

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